Functional Analysis

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Lecture 3

L^p-spaces for $p \in [1, \infty)$

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Let (Ω, Σ, μ) be a fixed measure space.

The **space of p-th power integrable functions**:

$$L^p(\mu) := \{x : \Omega \to \mathbb{F} \text{ measurable} : \int_\Omega |x(t)|^p \, d\mu < \infty\}$$

together with

$$(x + y)(t) := x(t) + y(t), \qquad (\lambda x)(t) := \lambda x(t) \qquad \left(\begin{array}{c} \text{pointwise} \\ \text{operations} \end{array} \right)$$

where $x, y \in L^p(\mu)$, $\lambda \in \mathbb{F}$, is a linear space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

There is a natural "norm" on $L^p(\mu)$ given by

$$\|x\|_p := \left(\int_\Omega |x(t)|^p \, d\mu \right)^{\frac{1}{p}}$$

Rem: $\|x\|_p = 0 \iff \mu(\{t \in \Omega : x(t) \neq 0\}) = 0 \overset{def}{\iff} x \overset{\mu\text{-a.e.}}{=} 0$

Theorem (Hölder's inequality)

For $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and for $x \in L^p(\mu)$, $y \in L^q(\mu)$

$$\|x \cdot y\|_1 \leqslant \|x\|_p \cdot \|y\|_q,$$

that is

$$\int_\Omega |xy|\, d\mu \leqslant \Big(\int_\Omega |x|^p\, d\mu\Big)^{\frac{1}{p}} \Big(\int_\Omega |y|^q\, d\mu\Big)^{\frac{1}{q}}$$

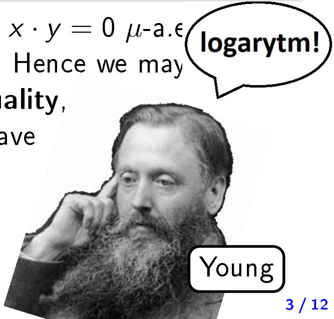
Equality holds $\iff$ $|x|^p$ and $|y|^q$ are linearly dependent μ-a.e.

(there are $\alpha, \beta \in \mathbb{R}$, $(\alpha, \beta) \neq (0,0)$, such that $\alpha|x|^p = \beta|y|^q$ μ-a.e.)

Proof: If $\|x\|_p = 0$, then $x = 0$ μ-a.e., whence $x \cdot y = 0$ μ-a.e
assertion holds trivially. Simlarily, for $\|y\|_q = 0$. Hence we may
that $\|x\|_p, \|y\|_q \neq 0$. We apply **Young's inequality**,
which says that for any numbers $a, b > 0$ we have

$$a \cdot b \leqslant \frac{1}{p} a^p + \frac{1}{q} b^q,$$

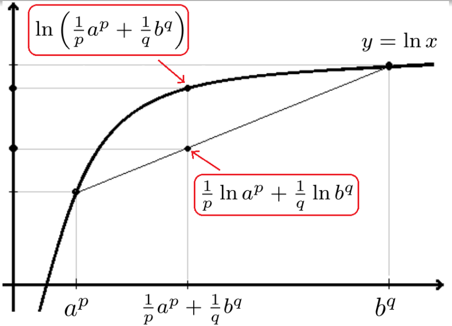
and follows from propeties of the logarithm:



$$
\begin{aligned}
\ln(ab) &= \ln a + \ln b \\
&= \tfrac{1}{p}\ln a^p + \tfrac{1}{q}\ln b^q \\
&\leqslant \ln\left(\tfrac{1}{p}a^p + \tfrac{1}{q}b^q\right),
\end{aligned}
$$

inequality holds by **concavity of logarithm**. By removing ln one gets **Young's inequality**.

Equality holds $\iff a^p = b^q$



Putting $a = \frac{|x(t)|}{\|x\|_p}$ and $b = \frac{|y(t)|}{\|y\|_q}$ we get

$$
\frac{|x(t)y(t)|}{\|x\|_p\|y\|_q} \leqslant \frac{1}{p}\cdot\frac{|x(t)|^p}{\|x\|_p^p} + \frac{1}{q}\cdot\frac{|y(t)|^q}{\|y\|_q^q}, \qquad \text{for all } t \in \Omega.
$$

Integrating the above inequality on both sides

$$
\frac{\int_\Omega |x(t)y(t)|\,d\mu}{\|x\|_p\|y\|_q} \leqslant \frac{1}{p}\cdot\frac{\int_\Omega |x(t)|^p\,d\mu}{\|x\|_p^p} + \frac{1}{q}\cdot\frac{\int_\Omega |y(t)|^q\,d\mu}{\|y\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1.
$$

Multiplying both sides by $\|x\|_p\|y\|_q$ we get $\|x\cdot y\|_1 \leqslant \|x\|_p\cdot\|y\|_q$.

Moreover, $\|x\cdot y\|_1 = \|x\|_p\cdot\|y\|_q \iff \frac{|x(t)|^p}{\|x\|_p^p} = \frac{|y(t)|^q}{\|y\|_q^q}$ μ-a.e.

$$\|x \cdot y\|_1 = \|x\|_p \cdot \|y\|_q \iff \|y\|_q^q \cdot |x|^p = \|x\|_p^p \cdot |y|^q \ \mu\text{-a.e.}$$
$$\iff |x|^p \text{ i } |y|^q \text{ are linearly dependent } \mu\text{-a.e}$$

„$\Leftarrow$" If $\alpha|x|^p = \beta|y|^q$ μ-a.e, then integrating $\alpha\|x\|_p^p = \beta\|y\|_q^q$.
Hence if $(\alpha, \beta) \neq (0, 0)$, then $\|y\|_q^q \cdot |x|^p = \|x\|_p^p \cdot |y|^q$ μ-a.e. ■

Theorem (Minkowski's inequality)

For any $p \geqslant 1$ and $x, y \in L^p(\mu)$

$$\|x + y\|_p \leqslant \|x\|_p + \|y\|_p.$$

Minkowski

Proof: For $p = 1$ the proof is easy:

$$\|x + y\|_1 = \int_\Omega |x + y| \, d\mu \overset{\text{trian. ineq.}}{\leqslant} \int_\Omega |x| + |y| \, d\mu = \int_\Omega |x| \, d\mu + \int_\Omega |y| \, d\mu$$

$$= \|x\|_1 + \|y\|_1.$$

Assume than that $p > 1$. Let $q := p/(p - 1)$, so $1/p + 1/q = 1$.
We will now apply **Hölder's inequality**!

$$\|x+y\|_p^p = \int_\Omega |x+y|^p \, d\mu = \int_\Omega |x+y| \cdot |x+y|^{p-1} \, d\mu$$

$$\overset{\text{trian. ineq.}}{\leqslant} \int_\Omega |x| \cdot |x+y|^{p-1} \, d\mu + \int_\Omega |y| \cdot |x+y|^{p-1} \, d\mu$$

$$\overset{\text{Hölder x2}}{\leqslant} \|x\|_p \cdot \left(\int_\Omega |x+y|^{q(p-1)} d\mu\right)^{\frac{1}{q}}$$

$$+ \|y\|_p \cdot \left(\int_\Omega |x+y|^{q(p-1)} d\mu\right)^{\frac{1}{q}}$$

$$\overset{q(p-1)=p}{=} \|x\|_p \cdot \|x+y\|_p^{p/q} + \|y\|_p \cdot \|x+y\|_p^{p/q}$$

$$= (\|x\|_p + \|y\|_p) \cdot \|x+y\|_p^{p/q}.$$

Divididing both sides by $\|x+y\|_p^{p/q}$ and using that $p - p/q = 1$ we get

$$\|x+y\|_p = \|x+y\|_p^{p-p/q} = \frac{\|x+y\|_p^p}{\|x+y\|_p^{p/q}} \leqslant \|x\|_p + \|y\|_p. \qquad \blacksquare$$

Convention:

In the space $L^p(\mu)$ we identify functions that are equal μ-a.e.
(formally, the elements of $L^p(\mu)$ are equivalence classes for $y \overset{\mu\text{-a.e}}{=} x$).
Hence $(L^p(\mu), \|\cdot\|_p)$ is a normed space!

Thm. $L^p(\mu)$ is a Banach space for any $p \in [1, \infty)$.

Proof: Let $\{x_n\}_{n=1}^{\infty} \subseteq L^p(\mu)$ be Cauchy. By passing to a subsequence we may assume that $\|x_n - x_m\|_p \leqslant \frac{1}{4^n}$ for $m \geqslant n$.

Lem. Lecture 1

We show that the set

$$A := \{t \in \Omega : \forall_N \exists_{n>N} \, |x_n(t) - x_{n+1}(t)| \geqslant 1/2^n\}$$

has measure zero and that $\{x_n\}_{n=1}^{\infty}$ is pointwise convergent on $\Omega \setminus A$.

Notice that $A = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n$, where $A_n := \{t : |x_n(t) - x_{n+1}(t)| \geqslant \frac{1}{2^n}\}$.

Moreover

$$\frac{1}{2^{np}} \mu(A_n) \leqslant \int_{A_n} \underbrace{|x_n - x_{n+1}|^p}_{\geqslant (1/2^n)^p} \, d\mu \leqslant \|x_n - x_{n+1}\|_p^p \leqslant \frac{1}{4^{np}},$$

whence $\mu(A_n) \leqslant \frac{1}{2^{np}}$. Hence

$$\mu(A) \leqslant \mu\Big(\bigcup_{n=N}^{\infty} A_n\Big) \leqslant \sum_{N=n}^{\infty} \mu(A_n) \leqslant \sum_{n=N}^{\infty} \frac{1}{2^{np}} \longrightarrow 0, \quad \text{when } N \to \infty$$

tail of the covergent series

Thus $\mu(A) = 0$.

$$t \in \Omega \setminus A \iff \exists_N \forall_{n \geqslant N} |x_n(t) - x_{n+1}(t)| < \tfrac{1}{2^n}$$

$$\implies \exists_N \forall_{m \geqslant n \geqslant N} |x_n(t) - x_m(t)| < \sum_{k=n}^{m-1} \frac{1}{2^k} \xrightarrow{n,m\to\infty} 0.$$

Hence for $t \in \Omega \setminus A$ the sequence $\{x_n(t)\}_{n=1}^{\infty}$ is Cauchy, and therefore convergent. Put $x(t) := \lim_{n\to\infty} x_n(t)$, when $t \in \Omega \setminus A$, and $x(t) = 0$, when $t \in A$. Then $x_n \xrightarrow{\mu\text{-a.e.}} x$.

$$t \in \Omega \setminus A \iff \exists_N \forall_{n \geqslant N} |x_n(t) - x_{n+1}(t)| < \frac{1}{2^n}$$

$$\implies \exists_N \forall_{m \geqslant n \geqslant N} |x_n(t) - x_m(t)| < \sum_{k=n}^{m-1} \frac{1}{2^k} \xrightarrow{n,m\to\infty} 0.$$

Hence for $t \in \Omega \setminus A$ the sequence $\{x_n(t)\}_{n=1}^{\infty}$ is Cauchy, and therefore convergent. Put $x(t) := \lim_{n\to\infty} x_n(t)$, when $t \in \Omega \setminus A$, and $x(t) = 0$, when $t \in A$. Then $x_n \xrightarrow{\mu\text{-a.e.}} x$. We show the convergence in norm:

$$\|x - x_n\|_p^p = \int_\Omega |x(t) - x_n(t)|^p \, d\mu = \int_{\Omega\setminus A} \lim_{m\to\infty} |x_m(t) - x_n(t)|^p \, d\mu$$

$$\overset{\text{Fatou}}{\leqslant} \liminf_{m\to\infty} \int_\Omega |x_m - x_n|^p \, d\mu \leqslant \sup_{m\geqslant n} \|x_n - x_m\|_p^p \leqslant (1/4)^{pn} \longrightarrow 0.$$

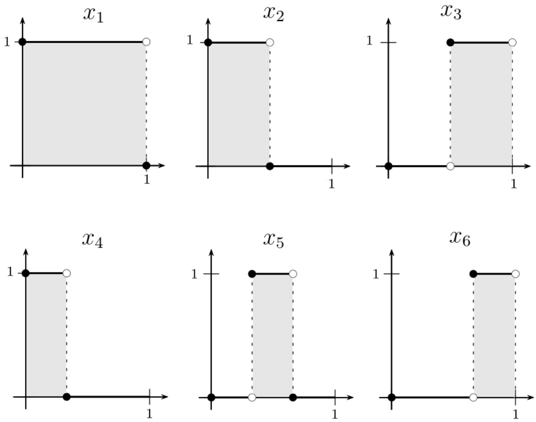
Hence $x_n \xrightarrow{\|\cdot\|_p} x$. As $\|x\|_p \leqslant \|x - x_n\|_p + \|x_n\|_p < \infty$, $x \in L^p(\mu)$. ∎

Remark. It follows from the proof above that

$$x_n \xrightarrow{\|\cdot\|_p} x \implies \exists_{\{x_{n_k}\}_{k=1}^{\infty}} x_{n_k} \xrightarrow{\mu\text{-a.e}} x.$$

However, in general $x_n \xrightarrow{\|\cdot\|_p} x \not\Longrightarrow x_n \xrightarrow{\mu\text{-a.e}} x$.

Ex. (wandering hump) On the space $L^p[0,1] := L^p(\lambda)$, where λ is the length on $[0,1]$ let's set k-element sequences $x_i^{(k)} = \mathbb{1}_{[\frac{i-1}{k},\frac{i}{k})}$, $i = 1, \ldots, k$, $k \in \mathbb{N}$, into one sequence $\{x_n\}_{n=1}^{\infty}$:



$$x_1 = \mathbb{1}_{[0,1)},\ x_2 = \mathbb{1}_{[0,\frac{1}{2})},$$
$$x_3 = \mathbb{1}_{[\frac{1}{2},1)},\ x_4 = \mathbb{1}_{[0,\frac{1}{3})},$$
$$x_5 = \mathbb{1}_{[\frac{1}{3},\frac{2}{3})},\ x_6 = \mathbb{1}_{[\frac{2}{3},1)}$$
$$x_7 = \mathbb{1}_{[0,\frac{1}{4})},\ x_8 = \mathbb{1}_{[\frac{1}{4},\frac{1}{2})}$$
$$x_9 = \mathbb{1}_{[\frac{1}{2},\frac{3}{4})},\ x_8 = \mathbb{1}_{[\frac{3}{4},1)}$$
$$\ldots$$

Then $x_n \xrightarrow{\|\cdot\|_p} 0$, because $\|x_i^{(k)}\|_p = (\int_{[0,1]} \mathbb{1}_{[\frac{i-1}{k},\frac{i}{k})}\, d\lambda)^{\frac{1}{p}} = (1/k)^{1/p} \to 0$ when $k \to \infty$. But for any $t \in [0,1)$ the sequence $\{x_n(t)\}_{n=1}^{\infty}$ is divergent (it has two limit points 0 and 1). Thus $x_n \overset{\mu\text{-a.e}}{\not\longrightarrow} 0$.

Integral over the counting measure is the sum!
Sequences are functions on $\mathbb{N}$ or $\{1, ..., n\}$!

Ex. If $\Omega = \mathbb{N}$ and μ is the counting measure, then $L^p(\mu)$, $p \in [1, \infty)$, is **the space of sequences summable in the p-th power**:

$$\ell^p := \left\{ x = (x(1), ..., x(n), ...) \in \mathbb{F}^{\mathbb{N}} : \sum_{k=1}^{\infty} |x(k)|^p < \infty \right\}$$

with coordinate-wise operations and the norm

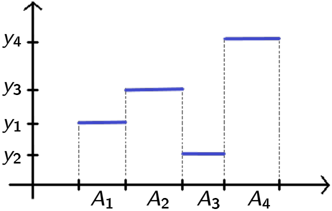
$$\|x\|_p := \left(\sum_{k=1}^{\infty} |x(k)|^p \right)^{\frac{1}{p}}$$

Ex. If $\Omega = \{1, ..., n\}$ and μ counting measure, then $L^p(\mu) \cong \mathbb{F}^n$ is the n-th dimensional Banach space with the norm

$$\|x\|_p := \left(\sum_{k=1}^{n} |x(k)|^p \right)^{\frac{1}{p}}$$

Indicator functions of sets with finite measure span the linear **space of integrable simple functions**

$$\mathcal{E}(\mu) := \operatorname{span}\{\mathbb{1}_{A_k} : A_k \in \Sigma, \mu(A_k) < \infty\}$$
$$= \left\{ \sum_{k=1}^{n} y_k \mathbb{1}_{A_k} : y_k \in \mathbb{F}, \mu(A_k) < \infty \right\}$$



Prop. For each $p \in [1, +\infty)$, $\mathcal{E}(\mu)$ is a dense subspace of $L^p(\mu)$. Hence $L^p(\mu) = \overline{\mathcal{E}(\mu)}^{\|\cdot\|_p}$ is the completion of $\mathcal{E}(\mu)$ in the norm

$$\|x\|_p := \left(\int_\Omega |x(t)|^p \, d\mu \right)^{\frac{1}{p}}$$

Proof: For $x \in L^p(\mu)$ there is $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{E}(\mu)$ with $|x_n| \leqslant |x|$ such that $x_n \to x$ pointwise. Since $|x - x_n|^p \leqslant 2|x|^p \in L^1(\mu)$

$$\|x - x_n\|_p^p = \int_\Omega |x(t) - x_n(t)|^p \, d\mu \xrightarrow{\text{dominated convergence}} 0. \qquad \blacksquare$$